

# LIPSCHITZ NORMAL EMBEDDINGS IN THE SPACE OF MATRICES

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**ABSTRACT.** The germ of an algebraic variety is naturally equipped with two different metrics up to bilipschitz equivalence. The inner metric and the outer metric. One calls a germ of a variety Lipschitz normally embedded if the two metrics are bilipschitz equivalent. In this article we prove Lipschitz normal embeddedness of some algebraic subsets of the space of matrices. These include the space  $m \times n$  matrices, symmetric matrices and skew-symmetric matrices of rank equal to a given number and their closures, and the upper triangular matrices with determinant 0. We also make a short discussion about generalizing these results to determinantal varieties in real and complex spaces.

## 1. INTRODUCTION

If  $(X, 0)$  is the germ of an algebraic (analytic) variety over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , then one can define two natural metrics on it. Both are defined by choosing an embedding of  $(X, 0)$  into  $(\mathbb{K}^N, 0)$ . The first is the *outer metric*, where the distance between two points  $x, y \in X$  is given by  $d_{out}(x, y) := \|x - y\|_{\mathbb{K}^N}$ , i.e. the restriction of the Euclidean metric to  $(X, 0)$ . The other is the *inner metric*, where the distance is defined as

$$(1) \quad d_{in}(x, y) := \inf_{\gamma} \{length_{\mathbb{K}^N}(\gamma) \mid \gamma: [0, 1] \rightarrow X \text{ rectifiable, } \gamma(0) = x, \gamma(1) = y\}.$$

Both of these metrics are independent of the choice of the embedding up to bilipschitz equivalence. The outer metric determines the inner metric, and it is clear that  $d_{out}(x, y) \leq d_{in}(x, y)$ . The other direction is in general not true, and one says that  $(X, 0)$  is *Lipschitz normally embedded* if the inner and outer metrics are bilipschitz equivalent. *Bilipschitz geometry* is the study of the bilipschitz equivalence classes of these two metrics.

The study of bilipschitz geometry of complex spaces started with Pham and Teissier who studied the case of curves in [PT69]. It then lay dormant for long time until Birbrair and Fernandes began studying the case of complex surfaces [BF08]. Among important recent results are the complete classification of the inner metrics of surfaces by Birbrair, Neumann and Pichon [BNP14], the proof that Zariski equisingularity is equivalent to bilipschitz triviality in the case of surfaces by Neumann and Pichon [NP14b] and the proof that outer Lipschitz regularity implies smoothness by Birbrair, Fernandes, Lê and Sampaio [BFLS16].

Understanding the geometry of the model varieties in the space of matrices is an important step in understanding determinantal singularities in real and complex spaces. We will also give a brief discussion of this.

Determinantal singularities is also an area that has been around for a long time, that recently saw a lot of interest. They can be seen as a generalization of isolated complete intersections (ICIS for short), and the recent results have mainly been in the study of invariants coming from their deformation theory. In [GZÈ09] Ébeling

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and Gusein-Zade defined the index of a 1-form, and the Milnor number has been defined in various different ways by Ruas and da Silva Pereira [SRDSP14], Damon and Pike [DP14] and Nuño-Ballesteros, Oréface-Okamoto and Tomazella [NBOOT13]. Their deformation theory has also been studied by Gaffney and Rangachev [GR15] and Fröhbis-Krüger and Zach [FZ15].

In January 2016 Asuf Shachar asked the following question on Mathoverflow.org (<http://mathoverflow.net/questions/222162>): Is the Lie group  $\mathrm{GL}_n^+(\mathbb{R})$  Lipschitz normally embedded, where  $\mathrm{GL}_n^+(\mathbb{R})$  is the group of  $n \times n$  matrices with positive determinants. A positive answer was given by the first author and Katz, Katz and Liokumovich in [KKKL17]. They first prove it for  $X_{n-1}$  the set of  $n \times n$  matrices with rank  $n-1$  and for its closure  $\overline{X_{n-1}}$ , the set of matrices with determinant equal to zero. Then they replace the segments of the straight line between two points of  $\mathrm{GL}_n^+(\mathbb{R})$  that passes through  $\mathrm{GL}_n^-(\mathbb{R})$  with a curve arbitrarily close to  $\overline{X_{n-1}}$ . Their proof relies on topological arguments, and some results on conical stratifications of MacPherson and Procesi [MP98]. In this article we give an alternative proof relying only on linear algebra and simple trigonometry, which also works for  $m \times n$  matrices of rank equal to  $t \leq \min\{m, n\}$  and their closures. (A first version of this proof appeared in [PR16]). We also prove the Lipschitz normal embeddedness of the symmetric and skew-symmetric matrices of rank equal to a given  $t$  and their closures, the upper triangular matrices which have determinant 0, and the intersections with linear subspaces transversal to the rank stratification.

This article is organized as follows. In section 2 we discuss the basic notions of Lipschitz normal embeddings and give some results concerning when a space is Lipschitz normally embedded. In section 3 we describe the basic properties of the bilipschitz geometry of the spaces of matrices we consider. In section 4 we prove that the set  $X_t$  of matrices, symmetric matrices and skew-symmetric matrices of rank equal to a given  $t$  and their corresponding closures  $\overline{X_t}$  are Lipschitz normally embedded, and that the same is true if  $V$  is a linear subspace transverse to the rank stratification. We prove that the space of upper triangular matrices with determinant 0 is Lipschitz normally embedded in section 5. Finally in section 6 we discuss some of the difficulties to extend these results to the setting of general determinantal singularities.

## 2. PRELIMINARIES ON BILIPSCHITZ GEOMETRY

In this section we discuss some properties of Lipschitz normal embeddings.

**Definition 2.1.** We say that  $X$  is *Lipschitz normally embedded* if there exist  $K \geq 1$  such that for all  $x, y \in X$ ,

$$(2) \quad d_{in}(x, y) \leq K d_{out}(x, y).$$

We call a  $K$  that satisfies the inequality a *bilipschitz constant* of  $X$ .

A trivial example of a Lipschitz normally embedded set is  $\mathbb{C}^n$ . For an example of a space that is not Lipschitz normally embedded, consider the plane curve given by  $x^3 - y^2 = 0$ , then  $d_{out}((t^2, t^3), (t^2, -t^3)) = 2|t|^3$  but the  $d_{in}((t^2, t^3), (t^2, -t^3)) = 2|t|^2 + o(t^2)$ , this implies that  $\frac{d_{in}((t^2, t^3), (t^2, -t^3))}{d_{out}((t^2, t^3), (t^2, -t^3))}$  is unbounded as  $t \rightarrow 0$ , hence there cannot exist a  $K$  satisfying (2).

Pham and Teissier [PT69] show that in general the outer geometry of a complex plane curve is equivalent to its embedded topological type, and the inner geometry is equivalent to the abstract topological type. Hence a plane curve is Lipschitz normally embedded if and only if it is a union of smooth curves intersecting transversely. See also Fernandes [Fer03] and Neumann and Pichon [NP14a].

In the cases of higher dimension the question of which singularities are Lipschitz normally embedded becomes much more complicated. It is no longer only rather trivial singularities that are Lipschitz normally embedded, for example in the case of surfaces the second author together with Neumann and Pichon, shows that rational surface singularities are Lipschitz normally embedded if and only if they are minimal [NPP15]. As we will later see, singularities in the space of matrices give examples of non-trivial Lipschitz normally embedded singularities in arbitrary dimensions.

**Remark 2.2.** A couple of remarks about notation. Throughout the article  $\mathbb{K}$  will always denote  $\mathbb{R}$  and  $\mathbb{C}$ . We will often be talking about different inner distances of two points  $x, y \in \mathbb{K}^N$ , when we consider  $x, y$  as lying in different subspaces, hence  $d_{in}^V(x, y)$  is the inner distance between  $x$  and  $y$  measured using the inner metric on the subspace  $V \subset \mathbb{K}^N$ . When we are using different outer metrics we also denote the outer distance measured in  $V$  by  $d_{out}^V(x, y)$ .

First we explore the relationship between being Lipschitz normally embedded local and being it global.

**Definition 2.3.** A space  $X$  is *locally Lipschitz normally embedded* at  $x \in X$  if there is an open neighbourhood  $U$  of  $x$ , such that  $U$  is Lipschitz normally embedded. We say that  $X$  is locally Lipschitz normally embedded if this condition holds for all  $x \in X$ .

It is clear that being Lipschitz normally embedded implies being locally Lipschitz normally embedded. In the other direction we have:

**Proposition 2.4.** *Let  $X$  be a connected, compact locally Lipschitz normally embedded space. Then  $X$  is Lipschitz normally embedded.*

*Proof.* For each  $x \in X$  let  $U_x$  be a Lipschitz normally embedded neighbourhood of  $x$ , and let  $K_x$  be a bilipschitz constant. This implies that if  $y \in X$  is very close to  $x$ , then  $d_{in}(x, y) \leq K_x d_{out}(x, y)$ . Consider the map

$$f(x, y) := \frac{d_{in}(x, y)}{d_{out}(x, y)} : M \times M \rightarrow \mathbb{R}.$$

Let  $U \subset M \times M$  be a small open tubular neighbourhood of the diagonal  $\Delta$ . Then  $f$  is continuous on the compact set  $(M \times M) \setminus U$  and locally bounded at each point. Thus it is globally bounded on  $(M \times M) \setminus U$  and also on  $U$ .  $\square$

A simple consequence of this is the following.

**Corollary 2.5.** *Let  $M$  be a connected compact manifold, then  $M$  is Lipschitz normally embedded.*

We will next give some results about when spaces constructed from Lipschitz normally embedded spaces are themselves Lipschitz normally embedded. First is the case of product spaces.

**Proposition 2.6.** *Let  $X \subset \mathbb{K}^n$  and  $Y \subset \mathbb{K}^m$  and let  $Z = X \times Y \subset \mathbb{K}^{n+m}$ .  $Z$  is Lipschitz normally embedded if and only if  $X$  and  $Y$  are Lipschitz normally embedded.*

*Proof.* First we prove the “if” direction. Let  $(x_1, y_1), (x_2, y_2) \in X \times Y$ . We need to show that

$$d_{in}^{X \times Y}((x_1, y_1)(x_2, y_2)) \leq K d_{out}^{X \times Y}((x_1, y_1)(x_2, y_2)).$$

Let  $K_X$  be the constant such that  $d_{in}^X(a, b) \leq K_X d_{out}^X(a, b)$  for all  $a, b \in X$ , and let  $K_Y$  be the constant such that  $d_{in}^Y(a, b) \leq K_Y d_{out}^Y(a, b)$  for all  $a, b \in Y$ . We get,

using the triangle inequality, that

$$d_{in}^{X \times Y}((x_1, y_1)(x_2, y_2)) \leq d_{in}^{X \times Y}((x_1, y_1)(x_1, y_2)) + d_{in}^{X \times Y}((x_1, y_2)(x_2, y_2)).$$

Now the points  $(x_1, y_1)$  and  $(x_1, y_2)$  both lie in the slice  $\{x_1\} \times Y$  and hence  $d_{in}^{X \times Y}((x_1, y_1)(x_1, y_2)) \leq d_{in}^Y(y_1, y_2)$  and likewise we have  $d_{in}^{X \times Y}((x_1, y_2)(x_2, y_2)) \leq d_{in}^X(x_1, x_2)$ . This then implies that

$$d_{in}^{X \times Y}((x_1, y_1)(x_2, y_2)) \leq K_Y d_{out}^Y(y_1, y_2) + K_X d_{out}^X(x_1, x_2),$$

where we use that  $X$  and  $Y$  are Lipschitz normally embedded. Now it is clear that  $d_{out}^{X \times Y}((x_1, y_1)(x_1, y_2)) = d_{out}^Y(y_1, y_2)$  and  $d_{out}^{X \times Y}((x_1, y_2)(x_2, y_2)) = d_{out}^X(x_1, x_2)$ . Also, since

$$d_{out}^{X \times Y}((x_1, y_1)(x_2, y_2))^2 = d_{out}^Y(y_1, y_2)^2 + d_{out}^X(x_1, x_2)^2$$

by definition of the product metric, we have that

$$d_{out}^{X \times Y}((x_1, y_1)(x_1, y_2)) \leq d_{out}^{X \times Y}((x_1, y_1)(x_2, y_2)) \text{ and}$$

$$d_{out}^{X \times Y}((x_1, y_2)(x_2, y_2)) \leq d_{out}^{X \times Y}((x_1, y_1)(x_2, y_2)).$$

It then follows that

$$d_{in}^{X \times Y}((x_1, y_1)(x_2, y_2)) \leq (K_Y + K_X) d_{out}^{X \times Y}((x_1, y_1)(x_2, y_2)).$$

For the other direction, let  $p, q \in X$  and consider any path  $\gamma: [0, 1] \rightarrow Z$  such that  $\gamma(0) = (p, 0)$  and  $\gamma(1) = (q, 0)$ . Now  $\gamma(t) = (\gamma_X(t), \gamma_Y(t))$  where  $\gamma_X: [0, 1] \rightarrow X$  and  $\gamma_Y: [0, 1] \rightarrow Y$  are paths and  $\gamma_X(0) = p$  and  $\gamma_X(1) = q$ . Now  $l(\gamma) \geq l(\gamma_X)$ , hence

$$d_{in}^X(p, q) \leq d_{in}^Z((p, 0), (q, 0)).$$

Since  $Z$  is Lipschitz normally embedded, there exist a  $K > 1$  such that  $d_{in}^Z(z_1, z_2) \leq K d_{out}^Z(z_1, z_2)$  for all  $z_1, z_2 \in Z$ . We also have that  $d_{out}^Z((p, 0), (q, 0)) = d_{out}^X(p, q)$ , since  $X$  is embedded in  $Z$  as  $X \times \{0\}$ . Hence

$$d_{in}^X(p, q) \leq K d_{out}^X(p, q).$$

The argument for  $Y$  being Lipschitz normally embedded is the same exchanging  $X$  with  $Y$ .  $\square$

**Proposition 2.7.** *Let  $X = \cup X_r \subset \mathbb{K}^n$  be a locally Lipschitz stratification (see Parusiński [Par93] Definition 1.1), and assume that  $X$  is Lipschitz normally embedded. Let  $V$  be a  $C^1$  manifold and let  $x \in V \cap X$ ,  $x \in X_r$ . Assume that there exist an open neighbourhood  $U$  of  $x$  such that for all  $y \in U \cap X$ ,  $y \in X_{r(y)}$ , we have that  $V$  is transverse to  $X_{r(y)}$  at  $y$ . Then  $V \cap X$  is locally Lipschitz normally embedded at  $x$ .*

*Proof.* Since  $V$  is transverse to  $X_{r(y)}$  at all  $y \in U \cap X$ , we can (maybe by shrinking  $U$ ) choose a map  $\rho: U \rightarrow X_r \cap U$  which is a proper submersion restricted to each stratum, such that  $\rho^{-1}(x) = V \cap U$ . By the Lipschitz isotopy lemma (Theorem 1.9 in [Par93]) there exist a bilipschitz trivialization  $\varphi: U \rightarrow U_S \times U_T$  of  $X$ , where  $U_S \subset \mathbb{K}^{\dim(X_r)}$  and  $U_T \subset \mathbb{K}^{\text{codim}(X_r)}$ , such that the following diagram commutes:

$$\begin{array}{ccc} X \cap U & \xrightarrow{\varphi} & \rho^{-1}(x) \times (X_r \cap U) \\ & \searrow \rho & \swarrow \pi \\ & X_r \cap U, & \end{array}$$

where  $\pi$  is just the projection to the second factor. Now  $\varphi$  is a bilipschitz map so  $\rho^{-1}(x) \times (X_r \cap U)$  is Lipschitz normally embedded since  $X \cap U$  is. Then we have by Proposition 2.6 that  $\rho^{-1}(x) = V \cap U$  is Lipschitz normally embedded.

If  $\dim(V) > \text{codim}(X_r)$  then if  $V_S = V \cap (X_r \cap U)$  we have that  $V \cap U$  is bilipschitz equivalent to  $V_T \times V_S$ . Now  $\dim(V_T) = \text{codim}(X_r)$ , so we can choose  $\rho$  as above such that  $\rho^{-1}(x) = V_T$ . Hence  $V_T \cap X$  is Lipschitz normally embedded and since  $V_S$  is  $C^1$  equivalent to  $\mathbb{K}^{\dim(V) - \text{codim}(X_r)}$  it is also Lipschitz normally embedded. Thus  $V \cap (X \cap U)$  is Lipschitz normally embedded by Proposition 2.6, since it is bilipschitz equivalent to  $(V_T \cap X) \times V_S$ .  $\square$

Another case we will need later is the case of cones.

**Proposition 2.8.** *Let  $X \subset \mathbb{K}^n$  be the cone over  $M \subset S$  with cone point the origin of  $\mathbb{K}^n$ , where  $S = S^{n-1}$  if  $\mathbb{K} = \mathbb{R}$  and  $S = S^{2n-1}$  if  $\mathbb{K} = \mathbb{C}$ . Then the following conditions hold:*

- (a) *If  $M$  is Lipschitz normally embedded then  $X$  is Lipschitz normally embedded.*
- (b) *If  $X$  is Lipschitz normally embedded and  $M$  is compact, then each of the connected components of  $M$  is Lipschitz normally embedded.*

*Proof.* We first prove (a). Since  $M$  is Lipschitz normally embedded with bilipschitz constant  $K_M$  the same is true for  $r \cdot M = rM$ , where  $r \in \mathbb{R}^+$ .

Let  $x, y \in X$ . We can assume that  $0 \leq \|x\| \leq \|y\|$ . If  $x = 0$  then  $d_{in}^X(x, y) = d_{out}(x, y)$  since the straight line through 0 and  $y$  is in  $X$  because  $X$  is conical.

If  $\|x\| = \|y\| = r$ , then  $x$  and  $y$  are both in  $rM$ , and hence

$$d_{in}^X(x, y) \leq d_{in}^{rM}(x, y) \leq K_M d_{out}(x, y).$$

Now if  $0 < \|x\| < \|y\|$  let  $y' = \frac{y}{\|y\|}\|x\|$ . Then  $d_{in}^X(y, y') = d_{out}(y, y')$  since they both lie on the same straight line through the origin. If  $r = \|x\|$ , then  $x, y' \in rM$ . Hence like before  $d_{in}^X(x, y') \leq K_M d_{out}(x, y')$ . Now  $y'$  is the point closest to  $y$  in  $rM$ . Hence all of  $rM$  lies on the other side of the affine hyperplane through  $y'$  orthogonal to the line  $\overline{yy'}$  from  $y$  to  $y'$ . Hence the angle between  $\overline{yy'}$  and the line  $\overline{y'x}$  between  $y'$  and  $x$  is more than  $\frac{\pi}{2}$ . Therefore, the Euclidean distance from  $y$  to  $x$  is larger than each of  $l(\overline{yy'})$  and  $l(\overline{y'x})$ . This gives us:

$$\begin{aligned} d_{in}^X(x, y) &\leq d_{in}^X(x, y') + d_{in}^X(y', y) \leq K_M d_{out}(x, y') + d_{out}(y', y) \\ &\leq (K_M + 1) d_{out}(x, y). \end{aligned}$$

To prove (b), assume that  $X$  is Lipschitz normally embedded, but a connected component  $M' \subset M$  is not Lipschitz normally embedded.

Since  $M'$  is compact we can assume that  $M'$  is not locally Lipschitz normally embedded at some point by Proposition 2.4. So let  $p \in M'$  be a point such that  $M'$  is not Lipschitz normally embedded in a small open neighbourhood  $U \subset M'$  of  $p$ . By Proposition 2.6 we have that  $U \times (-\varepsilon, \varepsilon)$  is not Lipschitz normally embedded, where  $0 < \varepsilon$  is much smaller than the distance from  $M$  to the origin. Now the quotient map from  $c: M \times [0, \infty) \rightarrow X$  induces an outer (and therefore also inner) bilipschitz equivalence of  $U \times (-\varepsilon, \varepsilon)$  with  $c(U \times (-\varepsilon, \varepsilon))$ . Since both  $U$  and  $\varepsilon$  can be chosen to be arbitrarily small, we have that there does not exist any small open neighbourhood of  $p \in X$  that is Lipschitz normally embedded, contradicting that  $X$  is Lipschitz normally embedded. Hence  $X$  being Lipschitz normally embedded implies that  $M'$  is Lipschitz normally embedded.  $\square$

**Remark 2.9.** (a) holds under the weaker hypothesis that  $M$  has a finite number of connected components each one being Lipschitz normally embedded, and such that for each pair of connected components  $X$  and  $Y$  we have  $d_{out}(X, Y) := \inf_{x \in X, y \in Y} \{d_{out}(x, y)\} > 0$ . If the number of connected components of  $M$  is not finite, then the result may fail as seen below. (In particular, it is not enough to ask that  $M$  is locally compact, locally path-connected and locally Lipschitz normal.)

- Let  $M = \bigcup_{n=1}^{\infty} \{e^{\frac{\pi i}{n}}\} \subset S^1$ . Thus  $M$  is non-connected, non-compact, but (trivially) locally path-connected, locally compact, locally Lipschitz normal. But  $Cone(M) \subset \mathbb{R}^2$  is not locally Lipschitz normal at the origin.

A consequence of Proposition 2.8 is the following.

**Corollary 2.10.** *Let  $(X, 0)$  be the germ of real or complex homogeneous variety with isolated singularity, then  $(X, 0)$  is Lipschitz normally embedded.*

We conclude this section with a useful lemma.

Let  $\varphi \circlearrowleft \mathbb{R}^N$  be a diffeomorphism in  $\mathbb{R}^N$ . For each  $x \in \mathbb{R}^N$  consider the Jacobian matrix  $\frac{d\varphi}{dx}$ , it is non-degenerate. Let  $\{\lambda_i(x)\}$  be its eigenvalues and fix  $\lambda_{max}(x) = \max\|\lambda_i(x)\|$ ,  $\lambda_{min}(x) = \min\|\lambda_i(x)\|$ . Define

$$\lambda_{max} := \sup_{x \in \mathbb{R}^N} \lambda_{max}(x) \leq \infty, \quad \lambda_{min} := \inf_{x \in \mathbb{R}^N} \lambda_{min}(x) \geq 0$$

**Lemma 2.11.** *For a diffeomorphism  $\varphi$  as above suppose  $0 < \lambda_{min}$  and  $\lambda_{max} < \infty$ . Let  $X \subset \mathbb{R}^N$  be any path-connected subset. Then for any  $x, y \in X$  the following holds:  $\lambda_{min} \cdot d_{in}^X(x, y) \leq d_{in}^{\varphi(X)}(\varphi(x), \varphi(y)) \leq \lambda_{max} \cdot d_{in}^X(x, y)$ .*

*Proof.* For fixed  $x, y \in X$  choose a rectifiable path  $\gamma \subset X$  connecting  $x, y$  and satisfying:  $length(\gamma) < d_{in}^X(x, y) + \varepsilon$ . Then  $\varphi(\gamma) \subset \varphi(X)$  connects  $\varphi(x), \varphi(y)$ . It remains to compare  $length(\gamma) = \int_0^1 \sqrt{\|\dot{\gamma}(t)\|^2} dt$  to

$$length(\varphi(\gamma)) = \int_0^1 \sqrt{\|\dot{\varphi}(\gamma(t))\|^2} dt = \int_0^1 \sqrt{\left\| \frac{d\varphi}{dx} \cdot (\dot{\gamma}(t)) \right\|^2} dt.$$

Note that  $\lambda_{min} \cdot \|\dot{\gamma}(t)\| \leq \left\| \frac{d\varphi}{dx} \cdot (\dot{\gamma}(t)) \right\| \leq \lambda_{max} \cdot \|\dot{\gamma}(t)\|$ . Thus the bounds follow.  $\square$

### 3. GEOMETRY IN THE SPACE OF MATRICES

Let  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  and take the vector space of  $m \times n$  matrices over  $\mathbb{K}$ ,  $Mat_{m \times n}(\mathbb{K})$ ,  $1 \leq m \leq n$ . We use the standard inner product on  $Mat_{m \times n}(\mathbb{K})$ ,  $\langle A, B \rangle := trace(\overline{A}B^t)$ , and the corresponding metric on  $Mat_{m \times n}(\mathbb{K}) \approx \mathbb{K}^{mn}$ .

For any subset  $X \subseteq Mat_{m \times n}(\mathbb{K})$  consider the stratification by rank,  $X_r := X \cap \{A \in Mat_{m \times n}(\mathbb{K}) \mid rank(A) = r\}$ . The strata  $X_r$  are connected when  $\mathbb{K} = \mathbb{C}$ , however when  $\mathbb{K} = \mathbb{R}$  they may have various connected components.

Besides the outer metric,

$$d_{out}(A, B) = \sqrt{trace\left((A - B) \cdot \overline{(A - B)}^t\right)},$$

the sets  $X_r$  have the inner metric,  $d_{in}^{X_r}(A, B)$ , as defined in Equation (1) in the introduction. Similarly for the closures,  $\{\overline{X_r}\}$ , one has  $d_{in}^{\overline{X_r}}(A, B)$ .

Note that for some linear subspaces of  $Mat_{m \times n}(\mathbb{K})$  the rank stratification is not Lipschitz normally embedded as we will see in Example 4.5.

**3.1. The relevant group actions.** We use the action of two groups on  $Mat_{m \times n}(\mathbb{K})$  and on the strata  $\{X_r\}$ .

- Consider the group  $U(m) = \{V \mid V \cdot \overline{V}^t = \mathbb{1}_{m \times m}\} \subset Mat_{m \times m}(\mathbb{C})$  and similarly  $U(n) \subset Mat_{n \times n}(\mathbb{C})$ . Their product acts,  $U(m) \times U(n) \circlearrowleft Mat_{m \times n}(\mathbb{C})$ , by  $A \rightarrow V_l A V_r$ . This group action is isometric, because we have that  $\langle A, B \rangle \rightsquigarrow \langle V_l A V_r, V_l B V_r \rangle = trace(V_l A V_r \cdot \overline{(V_l B V_r)}^t) = \langle A, B \rangle$ . For  $\mathbb{K} = \mathbb{R}$  one takes the group  $O(n)$ .

The group  $U(n)$  is connected, thus if  $A \stackrel{U(n)}{\sim} B$  then there exists a path from  $A$  to  $B$ , given by the  $U(n)$ -action. The group  $O(n)$  has two connected components, in some cases we use the component  $SO(n)$ .

Given a matrix  $A \in X_r$ , we can use the  $U(n)$ ,  $SO(n)$ - action to bring the left and right kernels,  $\ker_l(A)$ ,  $\ker_r(A)$ , to the form  $(\underbrace{0, \dots, 0}_r, *, \dots, *)$ .

With this assumption  $A$  becomes block-diagonal, hence we have that

$$A \sim \begin{bmatrix} A_{inv} & \mathbb{O} \\ \mathbb{O} & \mathbb{O}_{(m-r) \times (n-r)} \end{bmatrix},$$

here  $A_{inv} \in \text{Mat}_{r \times r}(\mathbb{K})$  is invertible.

- Consider the group  $\text{GL}_m \subset \text{Mat}_{m \times m}(\mathbb{K})$  and similarly  $\text{GL}_n \subset \text{Mat}_{n \times n}(\mathbb{K})$ . The product acts,  $\text{GL}_m \times \text{GL}_n \curvearrowright \text{Mat}_{m \times n}(\mathbb{K})$ , by  $A \rightarrow V_l A V_r$ . This group action is not isometric. However, for any fixed pair  $(V_l, V_r)$  the map  $A \rightarrow V_l A V_r$  is bilipschitz as we see in Corollary 3.1 below.

Moreover, the action preserves all the strata  $\{X_r\}$  and acts on them transitively, e.g. any matrix  $A \in X_r$  is equivalent to the canonical form,

$$A \sim \begin{bmatrix} \mathbb{I}_{r \times r} & \mathbb{O} \\ \mathbb{O} & \mathbb{O}_{(m-r) \times (n-r)} \end{bmatrix}.$$

Therefore the tangent space to any of  $X_r$ , at any point  $A$ , can be computed as the tangent space to the orbit of  $A$  under this group action.

The next result is an easy corollary of Lemma 2.11.

**Corollary 3.1.** *Let  $V \subset \text{Mat}_{m,n}(\mathbb{K})$  and  $(C_l, C_r) \in \text{GL}_m \times \text{GL}_n$ . Then the map  $A \rightarrow C_l A C_r$  is a bilipschitz map from  $V$  to  $W = C_l V C_r$ . In particular if  $A, B \in V$  satisfy  $d_{in}(A, B) \leq K d_{out}(A, B)$ , then  $d_{in}(C_l A C_r, C_l B C_r) \leq K d_{out}(C_l A C_r, C_l B C_r)$*

**3.2. Connected components of the strata.** We first remark that in both cases  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , the sets  $\bar{X}_r$  are connected for all  $r$  if  $X$  is a linear subspace.

Let  $\mathbb{K} = \mathbb{C}$  and  $X$  be one of  $\text{Mat}_{m \times n}(\mathbb{C})$ ,  $\text{Mat}_{m \times m}^{sym}(\mathbb{C})$ ,  $\text{Mat}_{m \times m}^{skew-sym}(\mathbb{C})$  or triangular matrices. Then all the strata  $X_r$  are connected. Indeed, they are all irreducible algebraic varieties and thus  $\dim_{\mathbb{C}}(\bar{X}_r) - \dim_{\mathbb{C}}(\bar{X}_{r-1}) \geq 1$ , i.e. the complements are of real codimension  $\geq 2$ .

For  $\mathbb{K} = \mathbb{R}$  the strata can have several connected components.

- Let  $X = \text{Mat}_{m \times n}(\mathbb{R})$ , for  $r = m = n$  we have the classical decomposition  $X_n = \text{GL}_n^+(\mathbb{R}) \amalg \text{GL}_n^-(\mathbb{R})$ . We prove that for  $r < m$  the strata  $X_r$  are connected. Indeed, given any  $A \in X_r$  bring it to the block-diagonal form,  $A \stackrel{SO(m) \times SO(n)}{\sim} A_{inv} \oplus \mathbb{O}$ , as above. Here  $A_{inv}$  is invertible and is defined up to  $SO(r) \times SO(r)$  transformation. Thus for any  $A, B \in X_r$  it is enough to connect  $A_{inv} \oplus \mathbb{O}$  to  $B_{inv} \oplus \mathbb{O}$ .

If  $\det(A_{inv} B_{inv}) > 0$  then the two matrices are connected just inside  $\text{GL}_r(\mathbb{R})$ . To address the case  $\det(A_{inv} B_{inv}) < 0$ , it is enough to connect  $A_{inv} \oplus \mathbb{O}$  to some  $\tilde{A}_{inv} \oplus \mathbb{O}$ , with  $\det(A_{inv} \tilde{A}_{inv}) < 0$ . We choose

$$\tilde{A}_{inv} = \begin{bmatrix} \mathbb{I}_{(r-1) \times (r-1)} & \mathbb{O} \\ \mathbb{O} & -1_{1 \times 1} \end{bmatrix} \cdot A_{inv}$$

and construct the needed path as follows. Choose any path  $(x(t), y(t))$  from  $(1, 0)$  to  $(-1, 0)$  inside  $\mathbb{R}^2 \setminus \{(0, 0)\}$ , e.g. a half-circle. Let  $V(t) \in \text{GL}_2^+(\mathbb{R})$  be a matrix family inducing this path, i.e.  $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} m = V(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $V(0) = \mathbb{I}$

and  $V(1) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ . Accordingly consider the path

$$A(t) = \begin{bmatrix} \mathbb{I}_{(r-1) \times (r-1)} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & V(t) & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} \end{bmatrix} \cdot A$$

By the construction  $A(t)$  lies inside  $X_r$  and connects  $A_{inv} \oplus \mathbb{O}$  to  $\tilde{A}_{inv} \oplus \mathbb{O}$ . For  $m < n$  all the strata are connected by the similar argument.

- For  $X = \text{Mat}_{n \times n}^{sym}(\mathbb{R})$  and any  $A \in X_r$  we have  $A \stackrel{SO(n)}{\sim} A_{inv} \oplus \mathbb{O}$ , as before. Then use  $SO(r)$  to diagonalize  $A_{inv}$ . The signs of the eigenvalues are preserved in continuous deformations inside  $X_r$ . Therefore the decomposition into the connected components is  $X_r = \coprod_{r_+ + r_- = r} \mathcal{U}_{r_+, r_-}$ , where  $\mathcal{U}_{r_+, r_-} \subset \text{Mat}_{r \times r}^{sym}(\mathbb{R})$  is the subset of matrices of signature  $(r_+, 0, r_-)$ .
- For  $X = \text{Mat}_{n \times n}^{skew-sym}(\mathbb{R})$  recall that the rank of a skew-symmetric matrix is always even, thus  $X_{2r+1} = \emptyset$  and we work only with  $X_{2r}$ . We prove that for  $2r < n$  the stratum  $X_{2r}$  is connected, while for  $n$ -even the stratum  $X_n$  has two connected components.

Suppose  $n$  is even, then the canonical form under the  $SO(n)$  action is  $\bigoplus_i \begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}$ , and one can bring any matrix to this form in a continuous way. (Because  $SO(n)$  is connected.) Furthermore, if all  $\lambda_i$  are non-zero, then we can assume  $\lambda_i > 0$  for  $i < n$ . Indeed, the negative  $\{\lambda_i\}$  can be turned into positive in pairs by the  $SO(n)$  transformation

$$\begin{aligned} & \begin{bmatrix} 1 & 0 & & \\ 0 & -1 & & \\ & & 1 & 0 \\ & & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & \lambda_i & 0 & 0 \\ -\lambda_i & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_j \\ 0 & 0 & -\lambda_j & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & & \\ 0 & -1 & & \\ & & 1 & 0 \\ & & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\lambda_i & 0 & 0 \\ \lambda_i & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_j \\ 0 & 0 & \lambda_j & 0 \end{bmatrix}. \end{aligned}$$

(Note again that  $SO(n)$  is connected.) Thus any canonical form is connected (inside  $X_n$ ) to either  $\bigoplus_i \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  or to  $(\bigoplus_i \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}) \oplus \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ . Finally we remark that the Pfaffian polynomial of a skew-symmetric matrix,  $Pf(A)$ , is continuous under deformations of  $A$  and  $Pf|_{\mathcal{U}_{even}} > 0$ , while  $Pf|_{\mathcal{U}_{odd}} < 0$ . Thus there are two connected components. Therefore  $X_n = \mathcal{U}_{even} \amalg \mathcal{U}_{odd}$ .

For  $X_{2r}$ , with  $2r < n$ , we first use the equivalence  $A \rightarrow V^t A V$ ,  $V \in SO(n)$ , to bring  $A$  to the form  $A_{inv} \oplus \mathbb{O}$ , with  $A_{inv} \in \text{Mat}_{2r \times 2r}^{skew-sym}(\mathbb{R})$ , as in paragraph 3.1. Then, as in the case of  $X_n$ , we bring  $A_{inv}$  to either  $\bigoplus_i \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  or  $(\bigoplus_i \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}) \oplus \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ . As  $2r < n$ , it remains to connect

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ to } \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This is done as in the case of  $\text{Mat}_{m \times n}(\mathbb{R})$ . We fix a matrix family,  $V(s) \in GL_2^+(\mathbb{R})$ , that connects  $(1, 0)$  to  $(-1, 0)$  and consider the path

$$(3) \quad \begin{bmatrix} 1 & \mathbb{O}_{1 \times 2} \\ \mathbb{O}_{2 \times 1} & V(s) \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & \mathbb{O} \\ \mathbb{O} & V(s) \end{bmatrix}^t = \begin{bmatrix} 0 & v_{11} & v_{21} \\ -v_{11} & 0 & 0 \\ -v_{21} & 0 & 0 \end{bmatrix}.$$



**3.3. The local structure of  $\overline{X_r}$  and "controlled path-connectedness".** In this section  $\mathbb{K} \in \mathbb{R}, \mathbb{C}$  and we always consider small neighbourhoods of spaces near some points. We freely use the germ notation, e.g.  $(\mathbb{K}^n, \mathbb{O})$  denotes a small neighbourhood of  $\mathbb{K}^n$  near the origin (i.e. near the zero matrix),  $(\overline{X_r}, A)$  denotes a small neighbourhood of the matrix  $A$  in  $\overline{X_r}$ , while  $T_A X_r$  denotes the tangent space of  $X_r$  at the point  $A \in X_r$ .

Sometimes to keep track of the size we denote the strata by  $X_r^{(m \times n)}$ .

**Lemma 3.2.** 1. Let  $X = \text{Mat}_{m \times n}(\mathbb{K})$ , fix some  $A \in \overline{X_r^{m \times n}}$ , with  $\text{rank}(A) = r_0 \leq r$ . Then

$$(\overline{X_r^{m \times n}}, A) \approx (\mathbb{K}^{mr_0 + nr_0 - r_0^2}, \mathbb{O}) \times (\overline{X_{r-r_0}^{(m-r_0) \times (n-r_0)}}, \mathbb{O}),$$

where the homeomorphism is almost metric preserving, i.e. the metric distortion can be assumed small if the germ representatives are small.

2. Similarly, for  $X = \text{Mat}_{m \times m}^{\text{sym}}(\mathbb{K})$  one has:

$$(\overline{X_r^{m \times m}}, A) \approx (\mathbb{K}^{mr_0 - \binom{r_0}{2}}, 0) \times (\overline{X_{r-r_0}^{(m-r_0) \times (m-r_0)}}, 0),$$

while for  $X = \text{Mat}_{m \times m}^{\text{skew-sym}}(\mathbb{K})$  one has:

$$(\overline{X_r^{m \times m}}, A) \approx (\mathbb{K}^{mr_0 - \binom{r_0+1}{2}}, 0) \times (\overline{X_{r-r_0}^{(m-r_0) \times (m-r_0)}}, 0).$$

*Proof.* (1). Using the linear isometries  $U(m) \times U(n)$  we can assume the left/right kernels of  $A$  in the form  $(\underbrace{0, \dots, 0}_{r_0}, *, \dots, *)$ , see paragraph 3.1. Therefore  $A =$

$A_{\text{inv}} \oplus \mathbb{O}_{(m-r_0) \times (n-r_0)}$ , here  $A_{\text{inv}} \in \text{Mat}_{r_0 \times r_0}(\mathbb{K})$  is invertible.

As the action  $\text{GL}_m \times \text{GL}_n \curvearrowright X_{r_0}$  is transitive (and smooth) we write down the tangent space  $T_A X_{r_0}$  as the tangent to the orbit using the calculation of the tangent space given in [ACGH85]:

$$T_A X_{r_0} = \text{Span}_{\mathbb{R}}(V_l A, A V_r)_{\substack{V_l \in \text{Mat}_{m \times m}(\mathbb{K}) \\ V_r \in \text{Mat}_{n \times n}(\mathbb{K})}} = \text{Span}_{\mathbb{K}} \left( \begin{bmatrix} * & * \\ * & \mathbb{O}_{(m-r_0) \times (n-r_0)} \end{bmatrix} \right)$$

As the stratum  $X_{r_0}$  is smooth (at any of its points), it can be rectified locally near  $A$  to its tangent space. Namely, there exists a homeomorphism,  $(\text{Mat}_{m \times n}(\mathbb{K}), A) \approx (\mathbb{K}^{mr_0 + nr_0 - r_0^2}, 0) \times (\mathbb{K}^{(m-r_0)(n-r_0)}, 0)$ , that sends  $(X_{r_0}, A)$  to  $(T_A X_{r_0}, 0) \times \{0\} = (\mathbb{K}^{mr_0 + nr_0 - r_0^2}, 0) \times \{0\}$ . This homeomorphism is assured by the implicit function theorem and can be chosen "almost metric preserving". More precisely, for any  $\varepsilon > 0$  the distortion of the distances will be less than  $\varepsilon$  provided we choose a small enough neighbourhood of  $A$  in  $X_{r_0}$ .

Restricting this homeomorphism to  $(\overline{X_r^{m \times n}}, A)$  we get the statement.

(2). The proof is essentially the same, just here one uses the action  $A \rightarrow V^t A V$ ,  $V \in U(n)$ .  $\square$

**Lemma 3.3.** 1. Let  $X = \text{Mat}_{m \times n}(\mathbb{K})$ . For any  $r \leq m \leq n$  the connected components of  $X_r$  are "controlled path-connected" near any point of  $\overline{X_r}$  in the following sense:

for any  $A \in \overline{X_r}$  and any  $\varepsilon > 0$  there exists  $\delta = \delta(A, \varepsilon)$  such that any points of the ball,  $P, Q \in \text{Ball}_{\delta}(A) \cap X_r$ , belonging to the same connected component of  $X_r$ , are connected (inside  $\text{Ball}_{\delta}(A) \cap X_r$ ) by a path of length  $< \varepsilon$ .

2. Similarly for the spaces of (skew-)symmetric matrices,  $X = \text{Mat}_{m \times m}^{\text{sym}}(\mathbb{K})$  or  $X = \text{Mat}_{m \times m}^{\text{skew-sym}}(\mathbb{K})$ , their strata are controlled path connected at any point.

*Proof.* (1). Let  $\text{rank}(A) = r_0 \leq r$ , by the last lemma there exist homeomorphisms as on the diagram.

$$\begin{array}{ccc}
(Mat_{m \times n}(\mathbb{K}), A) & \xrightarrow{\varphi} & (\mathbb{K}^{mr_0+nr_0-r_0^2}, \mathbb{O}) \times Mat_{(m-r_0) \times (n-r_0)}(\mathbb{K}) \\
\cup & & \cup \\
(\overline{X_r}, A) & \xrightarrow{\sim} & (\mathbb{K}^{mr_0+nr_0-r_0^2}, \mathbb{O}) \times (\overline{X_{r-r_0}^{(m-r_0) \times (n-r_0)}}), \mathbb{O}) \\
\cup & & \cup \\
(X_r, A) & \xrightarrow{\sim} & (\mathbb{K}^{mr_0+nr_0-r_0^2}, \mathbb{O}) \times (X_{r-r_0}^{(m-r_0) \times (n-r_0)}, \mathbb{O}).
\end{array}$$

Here in the last row we denote by  $(X_r, A)$  a small neighbourhood of  $X_r$  near  $A$ , even though  $A \notin X_r$ . Similarly for  $(X_{r-r_0}^{(m-r_0) \times (n-r_0)}, \mathbb{O})$ .

While  $\varphi$  does not preserve the distances, the distortions are small for small representative, therefore it is enough to prove the statement for the presentation on the right.

Write the coordinates of  $P, Q$  for this splitting,  $P \rightsquigarrow (P_1, P_2)$ ,  $Q \rightsquigarrow (Q_1, Q_2)$ , where  $P_1, Q_1 \in (\mathbb{K}^{mr_0+nr_0-r_0^2}, \mathbb{O})$ , while  $P_2, Q_2 \in (X_{r-r_0}^{(m-r_0) \times (n-r_0)}, \mathbb{O})$ .

Now take the paths  $(tP_1, P_2)$ ,  $(tQ_1, Q_2)$ , where  $t \in [0, 1]$ . Both paths lie inside  $(\mathbb{K}^{mr_0+nr_0-r_0^2}, \mathbb{O}) \times X_{r-r_0}^{(m-r_0) \times (n-r_0)}$ , thus their pre-images under  $\varphi$  lie inside  $X_r$ . And the lengths of both paths are small for  $\delta$  small. Therefore it remains to check the points  $(0, P_2)$ ,  $(0, Q_2)$ , i.e. to connect them by a short path that lies inside  $\{0\} \times X_{r-r_0}^{(m-r_0) \times (n-r_0)}$ .

By this transition we have reduced the problem from the case  $P, Q, A \in \overline{X_r^{m \times n}}$  to the case,  $P_2, Q_2, \mathbb{O} \in \overline{X_{r-r_0}^{(m-r_0) \times (n-r_0)}}$ . Note that  $0 < r - r_0 \leq m - r_0 \leq n - r_0$ . Note that  $P_2, Q_2$  still lie in the same connected component of  $X_{r-r_0}^{(m-r_0) \times (n-r_0)}$ , as the paths are in  $X_r$ .

Thus we have to prove:

for any  $\varepsilon > 0$  there exists  $\delta = \delta(\varepsilon)$  such that any points  $P, Q \in \text{Ball}_\delta(\mathbb{O}) \cap X_r \subset Mat_{m \times n}(\mathbb{K})$  are connected (inside  $X_r$ ) by a path of length  $< \varepsilon$ .

Alternatively: any point  $P \in \text{Ball}_\delta(\mathbb{O}) \cap X_r$  is connected to the special point  $\delta \cdot \mathbb{I}_{r \times r} \oplus \mathbb{O}_{(m-r) \times (n-r)}$  by a path of length  $< \varepsilon$ . And this later statement is immediate, apply the Gauss elimination procedure on rows and columns (by  $GL_m \times GL_n$ ) to get a path of bounded length.

(2). For the (skew-)symmetric case the proof is essentially the same, just the special point is now  $\delta \cdot \mathbb{I} \oplus (-\delta \cdot \mathbb{I}) \oplus \mathbb{O}_{(m-r) \times (n-r)}$  (the sizes depend on the signature) and instead of the Gauss elimination one uses the action  $A \rightarrow V^t A V$ .  $\square$

#### 4. LIPSCHITZ NORMALITY OF LINEAR SUBSPACES OF THE SPACE OF MATRICES

##### 4.1. Lipschitz normality for the closures $\overline{X_r}$ .

**Theorem 4.1.** Let  $\mathbb{K} \in \mathbb{R}, \mathbb{C}$  and  $X$  be one of the spaces  $Mat_{m \times n}(\mathbb{K})$ ,  $Mat_{n \times n}^{sym}(\mathbb{K})$ ,  $Mat_{n \times n}^{skew-sym}(\mathbb{K})$ . For any  $1 \leq r \leq m \leq n$  and  $A, B \in \overline{X_r}$  holds:  $\frac{d_{in}^{\overline{X_r}}(A, B)}{2\sqrt{2}} \leq d_{out}(A, B) \leq d_{in}^{\overline{X_r}}(A, B)$ .

*Proof.* The inequality on the right is immediate, we prove the one on the left.

We use the group action,  $U(m) \times U(n) \curvearrowright Mat_{m \times n}(\mathbb{K})$ , by  $A \rightarrow U A V$ , and  $U(n) \curvearrowright Mat_{n \times n}^{sym}(\mathbb{K})$ ,  $Mat_{n \times n}^{skew-sym}(\mathbb{K})$ , by  $A \rightarrow U^t A U$ , to bring  $A$  to the form

$$\begin{bmatrix} A_1 & \mathbb{O}_{r \times (n-r)} \\ \mathbb{O}_{(m-r) \times r} & \mathbb{O}_{(m-r) \times (n-r)} \end{bmatrix}.$$

Here  $A_1 \in Mat_{r \times r}(\mathbb{K})$ ,  $Mat_{r \times r}^{sym}(\mathbb{K})$ ,  $Mat_{r \times r}^{skew-sym}(\mathbb{K})$ . This action preserves  $X_r$ ,  $\overline{X_r}$  and the inner/outer distances. Therefore we can assume  $A$  in this form. Present

$B$  accordingly:  $\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$ . Then:

$$d_{out}(A, B) = \sqrt{\|A_1 - B_1\|^2 + \|B_2\|^2 + \|B_3\|^2 + \|B_4\|^2}$$

This is the distance along the straight segment. We will replace this straight segment by two parts, lying inside  $\overline{X_r}$ , whose total length is less than  $2d_{out}(A, B)$

Consider the path  $B(t) = \begin{bmatrix} B_1 & tB_2 \\ tB_3 & t^2B_4 \end{bmatrix}$  for  $t \in [0, 1]$ . We claim:  $B(t) \in \overline{X_r}$  for any  $t \in [0, 1]$ . Indeed, scaling a particular row/column does not increase the rank. And in the (skew-)symmetric case  $B(t)$  remains (skew-)symmetric.

Therefore we get an algebraic curve (inside  $\overline{X_r}$ ) that connects  $B = B(1)$  to  $B(0) = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}$ . The length of this path is:  $\int_0^1 \sqrt{\|B_2\|^2 + \|B_3\|^2 + 4t^2\|B_4\|^2} dt$ .

It remains to move from  $B(0)$  to  $A$ . In this case the straight segment  $\overline{B(0), A}$  lies inside  $\overline{X_r}$ . In total we get:

$$d_{in}^{\overline{X_r}}(A, B) \leq \int_0^1 \sqrt{\|B_2\|^2 + \|B_3\|^2 + 4t^2\|B_4\|^2} dt + \|A_1 - B_1\|.$$

Now we use the bounds

$$\int_0^1 \sqrt{\|B_2\|^2 + \|B_3\|^2 + 4t^2\|B_4\|^2} dt < 2\sqrt{\|B_2\|^2 + \|B_3\|^2 + \|B_4\|^2}$$

and  $x + y \leq \sqrt{2(x^2 + y^2)}$  to get:

$$\begin{aligned} d_{in}^{\overline{X_r}}(A, B) &< 2\sqrt{\|B_2\|^2 + \|B_3\|^2 + \|B_4\|^2} + \|A_1 - B_1\| \leq \\ &\leq 2\sqrt{2}\sqrt{\|A_1 - B_1\|^2 + \|B_2\|^2 + \|B_3\|^2 + \|B_4\|^2} = 2\sqrt{2} \cdot d_{out}(A, B). \end{aligned}$$

□

**Remark 4.2.** The constant  $2\sqrt{2}$  is certainly not the best one. For example, for  $X = \text{Mat}_{m \times n}(\mathbb{K})$  one can prove  $d_{in}^{\overline{X_r}}(A, B) \leq \sqrt{2}d_{out}(A, B)$  by first going along the straight segment  $\begin{bmatrix} B_1 & tB_2 \\ B_3 & tB_4 \end{bmatrix}$ , thus bringing  $B$  to the form  $\begin{bmatrix} B_1 & 0 \\ B_3 & 0 \end{bmatrix}$ , and then going along the straight segment  $\begin{bmatrix} tA_1 + (1-t)B_1 & 0 \\ (1-t)B_3 & 0 \end{bmatrix}$ .

Probably one can get even better bounds by using the appropriate metric on the Grassmannians of linear subspaces,  $Gr(\mathbb{K}^{m-r}, \mathbb{K}^m)$ ,  $Gr(\mathbb{K}^{n-r}, \mathbb{K}^n)$  or the Stiefel manifolds.

#### 4.2. Lipschitz normality for connected components of $X_r$ .

**Theorem 4.3.** Let  $\mathbb{K} \in \mathbb{R}, \mathbb{C}$  and  $X$  be one of the spaces  $\text{Mat}_{m \times n}(\mathbb{K})$ ,  $\text{Mat}_{n \times n}^{sym}(\mathbb{K})$ ,  $\text{Mat}_{n \times n}^{skew-sym}(\mathbb{K})$ . Suppose  $A, B$  belong to the same connected component of  $X_r$ , for some  $r \leq m$ . Then  $\frac{d_{in}^{X_r}(A, B)}{2\sqrt{2}} \leq d_{out}(A, B) \leq d_{in}^{X_r}(A, B)$ .

*Proof.* The inequality on the right is obvious, we prove the one on the left.

**Step 1.** (Reduction to the case of  $X_n$ .) As in the proof for  $\overline{X_r}$  we apply the action of  $U(m) \times U(n)$ , or  $U(n)$  in the (skew-)symmetric case, to bring  $A$  to the form  $\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$ . Accordingly  $B$  is brought to  $\begin{bmatrix} B_1 & * \\ * & * \end{bmatrix}$ . It might happen that  $\text{rank}(B_1) < r$ . To avoid this we can take arbitrarily small but generic deformation of  $B$  inside  $X_r$ . (For example, apply the group action that adds to the first  $r$  rows/columns a small but generic linear combination of all the other rows/columns.)

Now, as  $\text{rank}(B_1) = r$ , we can take the path  $B(t) = \begin{bmatrix} B_1 & t* \\ t* & t^2* \end{bmatrix}$ , as in the proof for  $\overline{X_r}$ . As in that proof the length of this path is less than  $2 \cdot \sqrt{\dots}$ .

We arrive to  $\begin{bmatrix} B_1 & \circ \\ \circ & \circ \end{bmatrix}$  and it remains to connect the matrices  $\begin{bmatrix} A_1 & \circ \\ \circ & \circ \end{bmatrix}, \begin{bmatrix} B_1 & \circ \\ \circ & \circ \end{bmatrix}$  inside  $X_r$  by a path of the total length  $\leq 2d_{out}(A_1, B_1) + \varepsilon$ . In particular, the initial question has been reduced to the stratum  $X_n$  of square matrices. Note also: as the path  $B(t)$  was fully inside  $X_r$ , the points  $A_1, B_1$  lie in the same connected component of  $X_r$ .

**Step 2.** Let  $A, B \in X_n$  where  $X = Mat_{n \times n}(\mathbb{K}), Mat_{n \times n}^{sym}(\mathbb{K})$  or  $Mat_{n \times n}^{skew-sym}(\mathbb{K})$ . (For skew-symmetric matrices this implies:  $n$  is even.)

Let  $\mathbb{K} = \mathbb{C}$  then all the strata are connected. Consider the straight segment  $[A, B] \subset X$ . Its endpoints lie in  $X_n$ , thus, by algebraicity of the strata, it intersects  $X_{n-1}$  in a finite number of points which is at most  $\deg(\overline{X_{n-1}})$ . Now, by the controlled path connectedness (Lemma 3.3), we can deform the path slightly at each of these point to push it into the stratum  $X_n$ . Hence we get a path inside  $X_n$  of length  $\leq d_{out}(A, B) + \varepsilon$ . Together with the path  $B(t)$  of step 1 this finishes the proof.

Suppose  $\mathbb{K} = \mathbb{R}$ , let  $\mathcal{U} \subset X_n$  be the prescribed connected component. We construct the needed path from  $A$  to  $B$  inside  $\mathcal{U}$ .

*The idea of construction.* In the case of  $\overline{X_r}$  the straight edge  $[A, B]$  was replaced by a straight edge  $[A, B(0)]$  and an algebraic curve from  $B(0)$  to  $B = B(1)$ , such that

$$length(A, B(0)) + length(B(0), B(1)) \leq 2\sqrt{2}d_{out}(A, B),$$

see the proof of Theorem 4.1. For  $\mathcal{U}$  we use the same idea, but we need to split into more paths to stay inside  $\mathcal{U}$ . In this way we produce several straight edges,  $[A, A_1], [A_1, A_2], \dots, [A_{k-1}, A_k], [A_k, B_k]$ , and algebraic curves,  $(B_k, B_{k-1}), (B_{k-1}, B_{k-2}), \dots, (B_1, B)$  such that

$$\begin{aligned} &length[A, A_1] + \dots + length[A_{k-1}, A_k] + length[A_k, B_k] + \\ &+ length(B_k, B_{k-1}) + \dots + length(B_1, B) < 2\sqrt{2}d_{out}(A, B) + \varepsilon. \end{aligned}$$

For  $X = Mat_{n \times n}(\mathbb{R})$  or  $Mat_{n \times n}^{skew-sym}(\mathbb{R})$  it is enough to take  $k = 1$ , but for  $Mat_{n \times n}^{sym}(\mathbb{R})$  the number  $k$  can be  $\lfloor \frac{n}{2} \rfloor$ . All these paths lie in  $\overline{\mathcal{U}}$  and each of them has some points in  $\mathcal{U}$ , thus (by algebraicity of  $\overline{X_r}$ ) each of the paths lies in  $\mathcal{U}$ , except for a finite number of points. At each such point we use the controlled-path-connectedness, lemma 3.3, to (slightly) deform the path into  $\mathcal{U}$ .

*The construction.* Fix  $A, B \in \mathcal{U} \subset X_n$ . The edge  $[A, B]$  does not necessarily lie inside  $\overline{\mathcal{U}}$ , thus (unlike the case  $\mathbb{K} = \mathbb{C}$ ) it cannot be pushed back into  $\mathcal{U}$  by a small deformation. Split the edge  $[A, B]$  into the intervals  $[A, A_1], [A_1, B_1], (B_1, B]$ , where  $[A, A_1] \subset \mathcal{U}, (B_1, B] \subset \mathcal{U}$  and  $A_1, B_1 \in \overline{\mathcal{U}} \setminus \mathcal{U}$ . Thus  $A_1, B_1 \in \overline{X_{n-1}}$ , and (after a small-but-generic deformation of  $A, B$  inside  $\mathcal{U}$ ) we can assume  $A_1, B_1 \in X_{n-1}$ . (In the case of  $X = Mat_{n \times n}^{skew-sym}(\mathbb{R})$  the rank drops by two, thus  $A_1, B_1 \in X_{n-2}$ .) As in the case of  $\overline{X_r}$ , we can assume (using the  $O(n) \times O(n)$  action)  $A_1 = \begin{bmatrix} \tilde{A}_1 & \circ \\ \circ & \circ \end{bmatrix}$ , where  $\tilde{A}_1$  is invertible. As in the case of  $\overline{X_r}$ , we take the algebraic curve  $B_1(t) = \begin{bmatrix} \tilde{B}_1 & t* \\ t* & t^2* \end{bmatrix}$ , for  $t \in [0, 1]$ . And we can assume  $\tilde{B}_1$  invertible, so this curve lies inside  $X_{n-1}$ . (For skew-symmetric matrices the curve lies inside  $X_{n-2}$ .)

It remains to connect  $A_1$  to  $B_1(0)$ , inside  $\overline{\mathcal{U}}$ , and to (slightly) deform this path into  $\mathcal{U}$ .

- The case  $\mathcal{U} = \text{GL}_n^+(\mathbb{R}) \subset X = \text{Mat}_{n \times n}(\mathbb{R})$ . Take the path

$$\begin{bmatrix} t\tilde{A}_1 + (1-t)\tilde{B}_1 & \mathbb{O} \\ \mathbb{O} & \varepsilon(t) \end{bmatrix},$$

with a continuous function  $[0, 1] \xrightarrow{\varepsilon(t)} \mathbb{R}$  that satisfies:

$$\begin{aligned} \varepsilon(t) \cdot \det(t\tilde{A}_1 + (1-t)\tilde{B}_1) &\geq 0, \\ \varepsilon(t) = 0 &\text{ iff } \det(t\tilde{A}_1 + (1-t)\tilde{B}_1) = 0 \\ \text{and } |\varepsilon(t)| &\ll 1 \text{ for any } t. \end{aligned}$$

This path lies inside  $\mathcal{U}$  except for a finite number of points, where  $\det(t\tilde{A}_1 + (1-t)\tilde{B}_1) = 0$ . Now, by the controlled path-connectedness, lemma 3.3, we can deform the path slightly at each of these points into  $\mathcal{U}$ . Thus we have connected (the small deformations of)  $A_1, B_1(0)$ , inside  $\mathcal{U}$ , by a path of total length at most  $d_{\text{out}}(A_1, B_1(0)) + \varepsilon$ . Together with  $B(t)$  this provides the needed path from  $A$  to  $B$  inside  $\mathcal{U}$ .

The case  $\mathcal{U} = \text{GL}_n^-(\mathbb{R})$  is similar.

- The case  $X = \text{Mat}_{n \times n}^{\text{skew-sym}}(\mathbb{R})$ , here  $\mathcal{U} \subset X_n$  is prescribed by the parity of the negative values among  $\{\lambda_i\}$ , see paragraph 3.2. Take the path

$$\begin{bmatrix} t\tilde{A}_1 + (1-t)\tilde{B}_1 & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & 0 & \varepsilon(t) \\ \mathbb{O} & -\varepsilon(t) & 0 \end{bmatrix},$$

where a continuous function  $[0, 1] \xrightarrow{\varepsilon(t)} \mathbb{R}$  satisfies:

$$\varepsilon(t) = 0 \text{ iff } \det(t\tilde{A}_1 + (1-t)\tilde{B}_1) = 0, \quad |\varepsilon(t)| \ll 1 \text{ for any } t,$$

and the sign of  $\varepsilon(t)$  is chosen in such a way that (whenever  $\det(t\tilde{A}_1 + (1-t)\tilde{B}_1) \neq 0$ ) the total number of negative values among  $\{\lambda_i\}$  is the one prescribed by  $\mathcal{U}$ . This path lies inside  $\mathcal{U}$ , except for a finite number of points where  $\det(t\tilde{A}_1 + (1-t)\tilde{B}_1) = 0$ . Now use the controlled path-connectedness and proceed as in the case  $\mathcal{U} = \text{GL}_n^+(\mathbb{R})$ .

- The case  $X = \text{Mat}_{n \times n}^{\text{sym}}(\mathbb{R})$ , here  $\mathcal{U} = \mathcal{U}_{n_+, n_-}$  = symmetric matrices of signature  $(n_+, 0, n_-)$ .

Suppose at all the points of the edge  $[A_1, B_1]$  holds:  $n_+(t) \leq n_+, n_-(t) \leq +n_-$ . Then we take the path

$$\begin{bmatrix} t\tilde{A}_1 + (1-t)\tilde{B}_1 & \mathbb{O} \\ \mathbb{O} & \varepsilon(t) \end{bmatrix},$$

and argue as above.

In general on the edge  $[\tilde{A}_1, \tilde{B}_1]$  there might occur points where one of the conditions  $n_+(t) \leq n_+, n_-(t) \leq n_-$  is violated. And this cannot be corrected by just one factor of  $\varepsilon(t)$ . Thus we use the reduction on the size of matrix. Split the edge  $[A_1, B_1]$  into  $[A_1, A_2], [A_2, B_2], [B_2, B_1]$ , where  $[A_1, A_2] \subset \overline{\mathcal{U}} \cap X_{n-1}$ ,  $[B_1, B_2] \subset \overline{\mathcal{U}} \cap X_{n-1}$ , and  $A_2, B_2 \in \overline{\mathcal{U}} \cap X_{n-2}$ . Push the paths  $[A_1, A_2], [B_1, B_2]$  slightly into  $\mathcal{U}$  by  $\varepsilon(t)$ -addition as before. Apply the general method to the edge  $[A_2, B_2]$ , i.e. by  $O(n) \times O(n)$  bring  $A_2$  to the canonical form, then degenerate the corresponding blocks in  $B_2$  to zero-blocks. The curve  $B_2(1) \rightsquigarrow B_2(0)$  is pushed into  $\mathcal{U}$

by  $\begin{bmatrix} \tilde{B}_2(t) & & \\ & \varepsilon_1(t) & 0 \\ & 0 & \varepsilon_2(t) \end{bmatrix}$ , where  $\varepsilon_i(t)$  are small corrections as above. Now

repeat the process for the edge  $[A_2, B_2(0)]$ , etc.

After at most  $\lfloor \frac{n}{2} \rfloor$  steps we get to some  $A_k, B_k$  of rank  $\leq \lfloor \frac{n}{2} \rfloor$ . For them we can take the "corrected" path

$$\begin{bmatrix} t\tilde{A}_1 + (1-t)\tilde{B}_1 & & & \\ & \varepsilon_1(t) & & \\ & & \ddots & \\ & & & \varepsilon_k(t) \end{bmatrix},$$

that lies in  $\mathcal{U}$ , except for a finite number of points. Now use the controlled path-connectedness.  $\square$

**4.3. Lipschitz normality for transversal intersections with  $\overline{X_r}$ .** For more general linear subspaces of the space of matrices we have the following result.

**Proposition 4.4.** *Let  $V \subset X = \text{Mat}_{m \times n}(\mathbb{K})$  be a linear subspace. Assume that  $V$  intersects  $X_r$  transversely for all  $r \neq 0$ . Then  $Y := V \cap \overline{X_r}$  is Lipschitz normally embedded.*

*Proof.* First notice that the stratification of  $\overline{X_r}$  is a locally Lipschitz stratification, since it is locally analytically trivial along any stratum. Also by Theorem 4.1  $\overline{X_r}$  is Lipschitz normally embedded. Since  $V$  linear then  $Y = V \cap \overline{X_r}$  is a cone over its link, with vertex 0.

Since  $V$  is transverse to all the strata of  $\overline{X_r}$  away from 0, then the stratification of  $\overline{X_r}$  induces a locally Lipschitz trivial stratification on  $Y \setminus \{0\}$ .

By Proposition 2.7  $Y \setminus \{0\}$  is locally Lipschitz normally embedded. The sphere  $S$  is transverse to all the strata of  $Y \setminus \{0\}$ , here  $S \subset \text{Mat}_{m \times n}$  is the sphere of radius 1 of real codimension 1 (i.e. if  $\mathbb{K} = \mathbb{R}$  then  $S = S^{mn-1}$  and if  $\mathbb{K} = \mathbb{C}$  then  $S = S^{2mn-1}$ ). Hence we can again use Proposition 2.7 to conclude that the link  $M := Y \cap S$  is locally Lipschitz normally embedded. Then by Proposition 2.4  $M$  is Lipschitz normally embedded, and  $Y$  is Lipschitz normally embedded by Proposition 2.8 since it is a cone over  $M$ .  $\square$

It is not true for all linear subspaces  $V$  that  $V \cap \overline{X_r}$  is Lipschitz normally embedded, as the next example shows.

**Example 4.5.** Let  $V \subset \text{Mat}_{3 \times 3}(\mathbb{C})$  be the linear subspace given as the image of the following map  $F: \mathbb{C}^3 \rightarrow \text{Mat}_{3 \times 3}(\mathbb{C})$ :

$$F(x, y, z) = \begin{pmatrix} x & 0 & z \\ y & x & 0 \\ 0 & y & x \end{pmatrix}.$$

Let  $Y := V \cap \overline{X_2}$ , where  $\overline{X_2}$  is the set of matrices in  $\text{Mat}_{3 \times 3}(\mathbb{C})$  with zero determinant, which is Lipschitz normally embedded by Theorem 4.1. Hence one would expect  $Y$  to be a nice space. On the other hand  $Y = V(x^3 - y^2z)$ , hence it is a family of cusps degeneration to a line. But  $Y$  being Lipschitz normally embedded would imply that the cusp  $x^3 - y^2 = 0$  is Lipschitz normally embedded by Proposition 2.6, since each non zero point on the  $z$ -axis has a neighbourhood which is a product of the cups and the  $z$ -axis. But the cusp is not Lipschitz normally embedded by the work of Pham and Teissier [PT69]. Hence  $Y$  is not Lipschitz normally embedded.

The proof of Proposition 4.4 uses the matrix structure of  $Mat_{m \times n}(\mathbb{K})$ , and the naive generalization to more general varieties does not hold.

- The statement "if  $X, Y \subset \mathbb{R}^N$  are two manifolds intersecting transversally then  $X \cap \overline{Y}$  is Lipschitz normally embedded" does not hold because of the obvious counterexample:  $Y = \{z = 0\} \subset \mathbb{R}^3$ ,  $\overline{X} = \{y^2 = x^3\} \subset \mathbb{R}^3$ ,  $X = \overline{X} \setminus \{x = 0 = y\}$ .
- The statement "if  $X, Y \subset \mathbb{R}^N$  are two manifolds intersecting transversally, with  $\overline{Y}$  Lipschitz normal then  $X \cap \overline{Y}$  is Lipschitz normally embedded" does not hold either. e.g. let  $\overline{Y} = \{x^2 + y^2 = z^k\} \subset \mathbb{R}^3$  and  $X = \{x = y\}$ . Then  $X \cap \overline{Y}$  is not Lipschitz normally embedded.
- Consider the embedding  $\mathbb{R}^2 \xrightarrow{j} X := Mat_{2 \times 2}(\mathbb{R})$  by

$$(x, y) \rightarrow \begin{pmatrix} y & x^{k-1} \\ x & y \end{pmatrix}.$$

Then  $j(\mathbb{R}^2)$  intersects transversally  $X_2$  and  $X_1$ . But  $j(\mathbb{R}^2) \cap \overline{X_1} \approx \{y^2 = x^k\} \subset \mathbb{R}^2$  is not bilipschitz normal at the origin. So the linearity of  $V$  is also important. We will in Section 6 look more on the case when  $V$  is not linear.

## 5. LIPSCHITZ NORMALITY OF COLLECTIONS OF AFFINE SUBSPACES IN $\mathbb{R}^N$

Fix a (possibly infinite) collection  $\{L_i\}$  of affine subspaces in  $\mathbb{R}^N$ , of (varying) positive dimensions. The union  $\cup L_i$  is not always Lipschitz normally embedded (of course we assume  $\cup L_i$  to be connected).

**Example 5.1.** The subset  $\{x(y^2 - 1) = 0\} \subset \mathbb{R}^2$  is not Lipschitz normally embedded because  $d_{out}((t, 1), (t, -1)) = 2$ , while  $d_{in}((t, 1), (t, -1)) = 2 + 2t$ .

In this example the collection contains two non-intersecting lines. We prove that in many cases this is the only obstruction to being Lipschitz normally embedded.

5.1. As a preparation we recall the definition of the angle between two (intersecting) affine subspaces  $L_i, L_j \subset \mathbb{R}^N$ . (All the metrics here are outer.)

- Suppose the intersection is just one point,  $L_i \cap L_j = \{0\}$ . Define the angle,  $\alpha_{L_i, L_j} \in [0, \frac{\pi}{2}]$ , via the theorem of cosines:

$$\cos(\alpha_{L_i, L_j}) = \sup_{\substack{x \in L_i \setminus \{0\} \\ y \in L_j \setminus \{0\}}} \frac{d^2(x, 0) + d^2(y, 0) - d^2(x, y)}{2d(x, 0)d(y, 0)}.$$

If  $L_i, L_j$  are lines this gives the classical definition, in particular it is independent on the choice of  $x, y$ .

- If  $\dim(L_i \cap L_j) > 0$  fix a point  $0 \in L_i \cap L_j$  and take the orthogonal complement at 0:  $(L_i \cap L_j) \oplus \mathcal{N} = \mathbb{R}^N$ . Then we define  $\alpha_{L_i, L_j} := \alpha_{(L_i \cap \mathcal{N}), (L_j \cap \mathcal{N})}$ . If  $\dim(L_i) = \dim(L_j)$  and  $\dim(L_i \cap L_j) = \dim(L_i) - 1$  then  $L_i \cap \mathcal{N}, L_j \cap \mathcal{N}$  are lines and we get the classical definition.

By its definition  $\alpha_{L_i, L_j} \in [0, \frac{\pi}{2}]$ .

**Lemma 5.2.** If  $L_i \not\subseteq L_j$  and  $L_j \not\subseteq L_i$  then  $\alpha_{L_i, L_j} \neq 0$ .

*Proof.* We can assume  $L_i \cap L_j$  is just one point and move this point to the origin. In the definition of  $\cos(\alpha_{L_i, L_j})$  apply the homogeneous scaling of  $\mathbb{R}^N$  to get:  $0 < \varepsilon \leq |x|, |y| \leq 1$ . As  $x \notin L_j$  and  $y \notin L_i$  the points  $x, 0, y$  are not on one line, thus:  $f(x, y) = \frac{d^2(x, 0) + d^2(y, 0) - d^2(x, y)}{2d(x, 0)d(y, 0)} < 1$ . As  $f(x, y)$  is a continuous function on

the compact domain,  $(L_i \cap \{\varepsilon \leq |x| \leq 1\}) \times (L_j \cap \{\varepsilon \leq |y| \leq 1\})$ , it attains its maximum. Therefore  $\cos(\alpha_{L_i, L_j}) = \sup_{\substack{x \in L_i \setminus \{0\} \\ y \in L_j \setminus \{0\}}} f(x, y) < 1$ .  $\square$

5.2. Now we use the angle  $\alpha_{L_i, L_j}$  to get the optimal Lipschitz constant.

**Proposition 5.3.** *Let  $X = \cup L_i \subset \mathbb{R}^N$  be the union of affine subspaces. Suppose  $L_i \not\subset L_j$  for any  $i \neq j$  and the subspaces intersect pairwise,  $L_i \cap L_j \neq \emptyset$ .*

(1) *For any  $x, y \in X$  holds:*

$$\frac{d_{in}^{(X)}(x, y)}{d_{out}(x, y)} \leq \sup_{i \neq j} \frac{1}{\sin(\frac{\alpha_{L_i, L_j}}{2})}.$$

(2) *If the collection is finite then the bound is asymptotically sharp, i.e. there exist sequences  $\{x_n\}, \{y_n\}$  satisfying:*

$$\frac{d_{in}^{(X)}(x_n, y_n)}{d_{out}(x_n, y_n)} \rightarrow \sup_{i \neq j} \frac{1}{\sin(\frac{\alpha_{L_i, L_j}}{2})}.$$

*Proof.* (1) Let  $x \in L_i, y \in L_j$ , the non-trivial case is  $i \neq j$ . Fix some  $0 \in L_i \cap L_j$  and use the theorem of sines for the triangle  $Conv(x, y, 0)$ :  $\frac{d(0, x)}{\sin(\alpha_y)} = \frac{d(0, y)}{\sin(\alpha_x)} = \frac{d(x, y)}{\sin(\alpha_0)}$ . Then one has:

$$\begin{aligned} d_{in}^{(X)}(x, y) &\leq d(x, 0) + d(0, y) = \frac{d_{out}(x, y)(\sin(\alpha_y) + \sin(\alpha_x))}{\sin(\alpha_0)} = \\ &= d_{out}(x, y) \frac{2\sin(\frac{\alpha_x + \alpha_y}{2})\cos(\frac{\alpha_x - \alpha_y}{2})}{\sin(\alpha_0)} \leq 2d_{out}(x, y) \frac{\cos(\frac{\alpha_0}{2})}{\sin(\alpha_0)}. \end{aligned}$$

Finally,  $\alpha_0 \geq \alpha_{L_i, L_j}$  hence  $d_{in}^{(X)}(x, y) \leq d_{out}(x, y) \frac{1}{\sin(\frac{\alpha_{L_i, L_j}}{2})}$ . This gives the bound.

(2) To prove the asymptotic sharpness note that for  $|x_n|, |y_n| \rightarrow \infty$ , the main contribution to  $d_{in}^{(X)}(x_n, y_n)$  comes from the paths inside  $L_i, L_j$ , while the possible corrections from other affine spaces become negligible.  $\square$

We remark that though the statement does not assume finiteness of the collection  $\{L_i\}$ , it is not very useful in the infinite case, as there  $\sup_{i \neq j} \frac{1}{\sin(\frac{\alpha_{L_i, L_j}}{2})}$  can easily go to infinity.

In this way one can produce many non-Cohen-Macaulay singularities which are still Lipschitz normally embedded.

**Example 5.4.** Suppose for some  $X \subset Mat_{m \times n}(\mathbb{K})$  the stratum  $\overline{X_r}$  consists of linear subspaces. (They all intersect as  $\mathbb{O}$  belongs to each of them.) Then  $\overline{X_r}$  is Lipschitz normally embedded. For example let  $X$  be the subspace of (upper/lower) triangular matrices in  $Mat_{m \times m}(\mathbb{R})$ , then  $\overline{X_{m-1}}$  is Lipschitz normally embedded. As all  $L_i$  are orthogonal in this case the optimal Lipschitz constant is  $\sqrt{2}$ .

## 6. THE CASE OF DETERMINANTAL SINGULARITIES

In this section we discuss Lipschitz normal embeddings of determinantal singularities. The spaces of matrices we worked with in the previous sections can be seen as special cases of determinantal singularities. In this section we assume that  $X = Mat_{m \times n}(\mathbb{K})$ , hence  $\overline{X_r}$  is the matrices of rank less than or equal to  $r$ . One could also work with  $Mat_{m \times n}^{sym}(\mathbb{K})$  or  $Mat_{m \times n}^{skew-sym}(\mathbb{K})$  but for simplicity we will restrict our discussion to  $Mat_{m \times n}(\mathbb{K})$ .



Let  $F: (\mathbb{K}^N, 0) \rightarrow (\text{Mat}_{mn}(\mathbb{K}), 0)$  be an analytic map germ. Then  $Y := F^{-1}(\overline{X}_r)$  is a *determinantal variety of type  $(m, n, r + 1)$*  if  $\text{codim}(Y) = \text{codim}(\overline{X}_r)$ , here we assume that  $r < \min\{m, n\}$ . Following Ébeling and Guseĭn-Zade [GZĖ09] a determinantal singularity  $Y = F^{-1}(\overline{X}_r)$  has an *essentially isolated singularity* at the origin (EIDS for short) if there is a neighbourhood  $U$  of the origin, such that  $F|_{U \setminus \{0\}}$  is transversal to the stratification of  $X_r$ . That is, for every  $x \in U \setminus \{0\}$ , rank of  $F(x) = s$ ,  $0 \leq s \leq r$ , then  $F$  is transversal to  $X_s$  at  $x$ . Any ICIS is an EIDS of type  $(m, 1, 1)$ .

With the notion of determinantal singularities Proposition 4.4 becomes the following:

**Theorem 6.1.** *Let  $Y$  is an EIDS defined by a linear map-germ  $F: \mathbb{K}^N \rightarrow \text{Mat}_{m \times n}$ , then  $Y$  is Lipschitz normally embedded.*

*Proof.* If  $F$  is injective then this is just a reformulation of Proposition 4.4. So assume that  $F$  is not injective, then we can decompose  $\mathbb{K}^N$  as  $\ker(F) \oplus V$ , where  $F$  induces an isomorphism from  $V$  to  $\text{Im}(F)$ . Hence  $Y = F^{-1}(\overline{X}_r)$  is isomorphic to  $\ker(F) \oplus (\text{Im}(F) \cap \overline{X}_r)$ . Now  $\ker(F)$  is a linear space and hence Lipschitz normally embedded and  $\text{Im}(F) \cap \overline{X}_r$  is Lipschitz normally embedded by Proposition 4.4, hence  $Y$  is Lipschitz normally embedded by Proposition 2.6.  $\square$

We can make a more general statement in Theorem 6.1. Take the group  $G = \mathcal{R} \times \mathcal{H}$  acting on the space of map-germs  $F: (\mathbb{K}^N, 0) \rightarrow (M_{m,n}(\mathbb{K}), 0)$  where  $\mathcal{R}$  is the group of germs of diffeomorphisms in  $(\mathbb{K}^N, 0)$  and  $\mathcal{H}$  is the group  $GL_m(\mathcal{O}_N) \times GL_n(\mathcal{O}_N)$ , given by invertible matrices with entries in  $(\mathcal{O}_N, 0)$  (see for instance Frühbis Krüger and Neumer's [FKN10]). As a consequence of Theorem 6.1 and Lemma 2.11 we can state the following:

**Corollary 6.2.** *If  $F: (\mathbb{K}^N, 0) \rightarrow (\text{Mat}_{m \times n}(\mathbb{K}), 0)$  is  $G$ -equivalent to a linear EIDS, then Theorem 6.1 holds.*

Whether a determinantal singularity is Lipschitz normally embedded is in general a more difficult question than for singularities in the space of matrices. One cannot in general expect a determinantal singularity to be Lipschitz normally embedded, the easiest way to see this is to note that all ICIS are determinantal, and that there are many ICIS that are not Lipschitz normally embedded. For example among the simple complex surface singularities  $A_n$ ,  $D_n$ ,  $E_6$ ,  $E_7$  and  $E_8$  only the  $A_n$ 's are Lipschitz normally embedded. Since the structure of determinantal singularities does not give us any new tools to study ICIS, we will probably not be able to say when an ICIS is Lipschitz normally embedded. Since  $F^{-1}(\overline{X}_0)$  is often an ICIS, we probably have to assume it is Lipschitz normally embedded to say anything about whether  $F^{-1}(\overline{X}_t)$  is Lipschitz normally embedded. But before we discuss such assumption further, we will see what went wrong in our Example 4.5 and give some more examples of determinantal singularities that are Lipschitz normally embedded and some that are not.

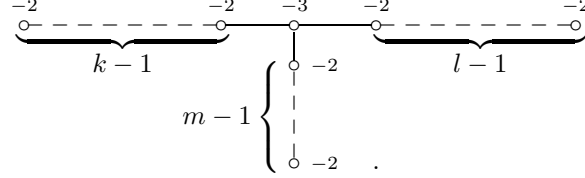
In Example 4.5,  $Y_0 := F^{-1}(\overline{X}_0)$  is a point and  $Y_1 := F^{-1}(\overline{X}_1)$  is a line, so both  $Y_0$  and  $Y_1$  are Lipschitz normally embedded. So it does not in general follows that if  $Y_i$  is Lipschitz normally embedded then  $Y_{i+1}$  is. Now the singularity in Example 4.5 is not an EIDS since  $F^{-1}(\overline{X}_1)$  does not have the expected dimension (the expected dimension is  $-1$ ). In the next example we will see that EIDS is not enough either.

**Example 6.3** (Simple Cohen-Macaulay codimensional 2 surface singularities). In [FKN10] Frühbis-Krüger and Neumer classify simple complex Cohen-Macaulay codimension 2 singularities. They are all EIDS of type  $(3, 2, 2)$ , and the surfaces correspond to the rational triple points classified by Tjurina [Tju68]. We will look

closer at two of such families. First we have the family given by the matrices:

$$\begin{pmatrix} z & y + w^l & w^m \\ w^k & y & x \end{pmatrix}.$$

This family corresponds to the family of triple points in [Tju68] called  $A_{k-1, l-1, m-1}$ . Tjurina shows that the dual resolution graph of their minimal resolution are:

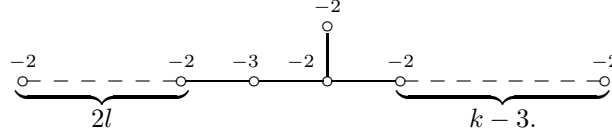


Using Remark 2.3 of [Spi90] we see that these singularities are minimal, and hence by the result of [NPP15] we get that they are Lipschitz normally embedded.

The second family is given by the matrices:

$$\begin{pmatrix} z & y + w^l & xw \\ w^k & x & y \end{pmatrix}.$$

Tjurina calls this family  $B_{2l, k-1}$  and give the dual resolution graphs of their minimal resolutions as:



Following Spivakovsky this is not a minimal singularity, and since it is rational according to Tjurina it is not Lipschitz normally embedded by the result of [NPP15].

These two families do not look very different but one is Lipschitz normally embedded and the other is not. We can do the same for all simple Cohen-Macaulay codimension 2 surfaces, and using the results in [NPP15], that rational surface singularities are Lipschitz normally embedded if and only if they are minimal, we get that only the family  $A_{l, k, m}$  is Lipschitz normally embedded. This is similar to the case of codimension 1, since only the  $A_n$  singularities are Lipschitz normally embedded among the simple singularities.

So as we see in Example 6.3 being an EIDS with singular set Lipschitz normally embedded, is not enough to ensure the variety is Lipschitz normally embedded. One should notice that the varieties in Example 4.5 and 6.3 are both defined by maps  $F: \mathbb{C}^N \rightarrow \text{Mat}_{m \times n}$  where  $N < mn$ . This means that one should think of the singularity as a section of  $\overline{X}_t$ , but being a subspace of a Lipschitz normally embedded space does not imply the Lipschitz normally embedded condition. If  $N \geq mn$  then one can think about the singularity being a fibration over  $\overline{X}_t$ , and as we saw in Proposition 2.6 products of Lipschitz normally embedded spaces are Lipschitz normally embedded. Now in this case  $Y_0 = F^{-1}(\overline{X}_0)$  is ICIS if  $Y$  is an EIDS, which means that we probably can not say anything general about whether it is Lipschitz normally embedded or not. So natural assumptions would be to assume that  $Y$  is an EIDS and that  $Y_0$  is Lipschitz normally embedded.

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